Exercise 96

A cup of hot chocolate has temperature 80°C in a room kept at 20°C. After half an hour the hot chocolate cools to 60°C.

(a) What is the temperature of the chocolate after another half hour?

(b) When will the chocolate have cooled to 40°C?

Solution

According to Newton's law of cooling, the governing equation for the temperature of the hot chocolate is

$$\frac{dT}{dt} = -k(T - 20),$$

where 20 is the surrounding temperature. To solve this differential equation, make the substitution $y = T - 20$. Then, differentiating both sides with respect to t, $\frac{dy}{dt} = \frac{dT}{dt}$.

$$\frac{dy}{dt} = -ky$$

Divide both sides by y.

$$\frac{1}{y}\frac{dy}{dt} = -k$$

Rewrite the left side as the derivative of a logarithm by using the chain rule.

$$\frac{d}{dt}\ln y = -k$$

The function you differentiate to get $-k$ is $-kt + C$, where C is any constant.

$$\ln y = -kt + C$$

Exponentiate both sides to solve for y.

$$e^{\ln y} = e^{-kt+C}$$

$$y(t) = e^C e^{-kt}$$

Use a new constant A for e^C.

$$y(t) = Ae^{-kt}$$

Now that the differential equation has been solved, change back to the original variable.

$$T - 20 = Ae^{-kt}$$

As a result, the general solution for the temperature is

$$T(t) = 20 + Ae^{-kt}.$$

Let $t = 0$ be the time at which the temperature is 80°C. Then $T(0) = 80$ and $T(0.5) = 60$.

$$\begin{cases} 80 = 20 + Ae^{-k(0)} \\ 60 = 20 + Ae^{-k(0.5)} \end{cases}$$

Solve the first equation for A,

$$A = 60,$$

and substitute this result into the second equation.

$$60 = 20 + 60e^{-0.5k}$$

$$40 = 60e^{-0.5k}$$

$$\frac{2}{3} = e^{-0.5k}$$

$$\ln\frac{2}{3} = \ln e^{-0.5k}$$

$$-\ln\frac{3}{2} = (-0.5k)\ln e$$

$$k = 2\ln\frac{3}{2}$$

Therefore, the temperature of the hot chocolate after t hours is

$$\begin{aligned} T(t) &= 20 + Ae^{-kt} \\ &= 20 + 60e^{-\left(2\ln\frac{3}{2}\right)t} \\ &= 20 + 60e^{\ln\left(\frac{3}{2}\right)^{-2t}} \\ &= 20 + 60\left(\frac{3}{2}\right)^{-2t}. \end{aligned}$$

Part (a)

The temperature of the chocolate after another half hour is

$$T(1) = 20 + 60\left(\frac{3}{2}\right)^{-2(1)} = \frac{140}{3} \approx 46.6667^\circ\text{C}.$$

Part (b)

To determine when the chocolate cools to 40°C, set $T(t) = 40$ and solve the equation for t.

$$T(t) = 40$$

$$20 + 60\left(\frac{3}{2}\right)^{-2t} = 40$$

$$60\left(\frac{3}{2}\right)^{-2t} = 20$$

$$\left(\frac{3}{2}\right)^{-2t} = \frac{1}{3}$$

$$\ln\left(\frac{3}{2}\right)^{-2t} = \ln\frac{1}{3}$$

$$(-2t)\ln\left(\frac{3}{2}\right) = -\ln 3$$

$$t = \frac{1}{2}\left(\frac{\ln 3}{\ln \frac{3}{2}}\right)$$

$$t \approx 1.35476 \text{ hours}$$

Convert 0.35476 hours to minutes.

$$0.35476 \text{ hours} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \approx 21.2853 \text{ minutes}$$

Convert 0.2853 minutes to seconds.

$$0.2853 \text{ minutes} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \approx 17.1203 \text{ seconds}$$

Therefore, the hot chocolate cools to 40°C about 1 hour and 21 minutes and 17 seconds after its temperature is 80°C.